Algebraic Number Theory

Exercise Sheet 13

Prof. Dr. Nikita Geldhauser
PD Dr. Maksim Zhykhovich Winter Semester 2024-25
30.01.2025

Exercise 1. Let x be the real number such that $x^3 - 2 = 0$ and K the field $\mathbb{Q}(x)$.
(1) Show that $X^3 - 2$ is irreducible in $\mathbb{Q}[X]$ and that $[K : \mathbb{Q}] = 3$. Find r_1 and r_2 for the field K.
(2) Let $z = a + bx + cx^2$ ($a, b, c \in \mathbb{Q}$) be an arbitrary element of K. Compute the trace of z over $\mathbb{Q}$.
(3) Let A be the ring of integers $\mathcal{O}_K$ of K and let $B \subset K$ be the ring $\mathbb{Z}[x]$ generated in K by x. Show that $B \subset A$ and that B is a free abelian group with basis $\{1, x, x^2\}$.
(4) Let $z = a + bx + cx^2$ ($a, b, c \in \mathbb{Q}$) be an integer in K. Use the computation of the traces of z, zx, zx^2 to show that $6A \subset B$.
(5) Show that B/xB is a field with two elements and conclude that the ideal xB is maximal in B.
(6) Let $a \in A$. Show the following: if $N_{\mathbb{Q}}^{K}(a)$ is a prime number in $\mathbb{Z}$, then the principal ideal (a) is prime in A. Deduce that xA is a prime ideal in A.
(7) Using the equality $2 = x^3$, compute $[A/xA : \mathbb{Z}/2]$, and give another proof that xA is a prime ideal in A.
(8) Show that $xB = B \cap xA$ and that $B/xB \to A/xA$ is an isomorphism.
(9) Using (8), conclude that $A = B + xA$ and then that $A = B + 2A$.
(10) Show that $3 = (x-1)(x+1)^3$ and that $x - 1$ is invertible in B. Proceed as in the questions (5)-(9) with $x + 1$ in place of x and 3 in place of 2, to show that $A = B + (x + 1)A$ and then that $A = B + 3A$.
(11) Using (9) and (10), show that $A = B + 6A$. Deduce from (4) that $A = B$.
(12) Let d_K be the absolute discriminant of K. Compute that $|d_K| = 27 \cdot 4$.
(13) Compute the class group $C(A)$.
Hint: Use the inequality from Korollar 9 (Chapter II) and estimate $(\frac{4}{\pi})^{r_2} \frac{n!}{n^n} \sqrt{|d_K|}$.
(14) Deduce from (13) that A is principal.